LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FOURTH SEMESTER – **APRIL 2023**

PMT 4504 – CALCULUS OF VARIATIONS AND INTEGRAL EQUATIONS

Date: 05-05-2023 Dept. No. Max. : 100 Marks

Time: 09:00 AM - 12:00 NOON

Answer ALL Questions:

1. a) Show that $y = (1+x^2)^{-\frac{3}{2}}$ is a solution of the integral equation, $y(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} y(t)dt$. **(5 marks)**

OR

b) Explain the different types of integral equation. **(5 marks)**

c) Transform $\frac{d^2y}{dx^2} + xy = 1$, with boundary conditions $y(0) = y(1) = 0$, into an integral equation. Also recover the boundary value from the integral equation obtained. **(15 marks)**

OR

d) Show that $y(x) = \sin\left(\frac{\pi x}{2}\right)$ is a solution of the Fredholm integral equation

$y(x) = \frac{x}{2} - \frac{\pi^2}{4}\int_0^1 k(x,t)y(t)dt$, where the kernel $k(x,t)$ is of the form

$$k(x,t) = \begin{cases} \frac{1}{2}x(2-t); 0 \le x < t \\ \frac{1}{2}t(2-x); t \le x \le 1 \end{cases}$$

(10 marks)

e) Find the integral equation of the initial value problem $\frac{d^2y}{dx^2} + y = 0; y(0) = 1, y'(0) = 0$. **(5 marks)**

2. a) Solve $y(x) = e^x + \lambda \int_0^1 2e^t e^x y(t)dt$. **(5 marks)**

OR

b) Find a non-trivial solution λ, from the integral equation $\emptyset(x) = \lambda \int_0^1 e^{t+x} \emptyset(t)dt$. **(5 marks)**

c) Determine the eigen values and eigen functions of the homogeneous integral equation equation $y(x) = \lambda \int_0^1 k(x,t)y(t)dt$ where the kernel $k(x,t)$ is of the form $k(x,t) = \begin{cases} t(x+1); 0 \le x \le t \\ x(t+1); t \le x \le 1 \end{cases}$. **(15 marks)**

OR

d) Find the eigen values and eigen functions of $y(x) = \lambda \int_0^{2\pi} sin(x+t)y(t)dt$. **(15 marks)**

3. a) Write the procedure to find the solution of Volterra integral equation using resolvent kernel. **(5 marks)**

OR

b) Find the solution of $y(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xty(t)dt$ by successive approximation method. **(5 marks)**

c) Solve $y(x) = x + \int_0^x (t-x)y(t)dt$ using resolvent kernel. **(15 marks)**

OR

d) Write the solution of Volterra integral equation of the second kind by successive approximations using Neumann series method. **(15 marks)**

4. a) Test for an extremum the functional $I[y(x)] = \int_0^1 (xy + y^2 - 2y^2 y')dx,\ y(0) = 1, y(1) = 2.$ **(5 marks)**

OR

b) Prove that the extremals of the functional $I[x(t), y(t)] = \int_{t_0}^{t_1} \left[(\dot{x}^2 + \dot{y}^2)^{1/2} + a^2(x\dot{y} - y\dot{x})\right]dt$ are circles. **(5 marks)**

c) Derive the necessary condition for the existence of extremal for the functional $I[y(x)] = \int_a^b F(x, y(x), y'(x))\,dx$ subject to the boundary conditions $y(a) = y_1, y(b) = y_2$ where y_1, y_2 are prescribed at the fixed boundary points *a*, *b* and $F(x, y(x), y'(x))$ is three times differentiable. Use the condition, to find the curve with fixed boundary points such that its rotation about the axis of abscissae give rise to a surface of revolution of minimum surface area. **(15 marks)**

OR

d) (i) Find the extremal of the functional $I[y_1(x), y_2(x)] = \int_a^b \left(2y_1y_2 - 2{y_1}^2 + {y_1'}^2 - {y_2'}^2\right)dx.$ **(10 marks)**

(ii) Find the Euler-Ostrogradsky equation for $I[u(x,y)] = \iint_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy$ where the values of *u* are prescribed on the boundary Γ of the domain *D*. **(5 marks)**

5. a) Derive Weirstrass function. **(5 marks)**

OR

b) Find the shortest distance between the parabola $x^2 = y$ and the straight line $x - y = 5$. **(5 marks)**

c) Discuss the Jacobi and Legendre conditions for extremum of the functional $I = \int_0^1 \left(\frac{x^2y'^2}{2} - 2xyy' + y\right)dx$, subject to the condition $u(0) = 0$ where $u = \delta y$. Also, derive the extremal satisfying $u(1) = \frac{1}{2}$ and emanating from $(0,1)$. **(15 marks)**

OR

d) Using only the basic necessary conditions $\delta I = 0$, find the curve on which an extremum of the functional $I(y(x)) = \int_0^{x_1} \frac{(1+y'^2)^{\frac{1}{2}}}{y}dx,\ y(0) = 0$ can be achieved if the second boundary point (x_1, y_1) can move along the circumference of the circle $(x-9)^2 + y^2 = 9$. **(15 marks)**

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